Self-Organization in Three-Dimensional Hydrodynamic Turbulence

G. Knorr, J. P. Lynov, and H. L. Pécseli

Association EURATOM – Risø National Laboratory, Optics and Fluid Dynamics Department,
P.O. Box 49, DK-4000 Roskilde, Denmark

The three-dimensional incompressible Euler equations are expanded in eigenflows of the curl operator, which represent positive and negative helicity flows in a particularly simple and convenient way. Four different basic types of interactions between eigenflows are found. Two represent an "inverse cascade", the interaction familiar from the two-dimensional Euler equations, in which only modes of the same sign of the helicity interact. The other two interactions mix positive and negative helicity modes. Only these interactions can transport all of the available energy to higher wave numbers. Initial conditions, which lead to the appearance of structures and self-organization, are discussed.

I. Introduction

There is no doubt that from an observational standpoint turbulent motion is often far away from a state of isotropic, homogeneous turbulence. Large-scale structures are observed, which are much longer lived than one would expect according to the picture of energy cascading to larger and larger wave numbers [1–2]. During the past two decades, structures in turbulence have been discussed widely [3].

Already in 1966, Freymuth [4] observes the growth of small disturbances in a separated laminar boundary layer with highly regular vortex motions in the nonlinear states of transition. Typically, two adjacent vortices rotate about a common axis and coalesce into a single structure. Further downstream the regular vortices give way to chaotic motion. Winant and Browandt [5] also observe adjacent vortices which amalgate into a single vortex of a larger scale. They suggest "pairing" is the principle mechanism by which shear layers grow. This may be associated with the noise, an important engineering application. Winant and Browandt [5] also observe entrainment of the fluid into regions of high vorticity.

Effects like the confluence of two vortices into a larger one are reminiscent of two-dimensional hydrodynamics, where under the condition of "negative temperature" the same processes occur. They have been studied analytically and numerically [6] and are well understood. An essential ingredient of the theory is that in two-dimensions (2 D) two invariants of the Euler equations exist, the energy and the enstrophy which are both positive definite [7]. They restrict the dynamics of the system in such a way that a so-called inverse cascade may result: in order for energy to migrate to higher wave numbers, the constancy of energy and enstrophy enforce also a migration of energy to lower wave numbers, which may result in large structures in configuration space.

In three-dimensional (3 D) hydrodynamics, however, the situation is completely different. Vortices may change their orientation in space, whereas in 2 D the vorticity vectors are always perpendicular to the plane, which contains the flow. In 3 D we have the so-called vortex stretching term, which vanishes identically in 2 D. Contrary to 2 D with positive definite quadratic invariants, there is only one positive definite quadratic invariant in 3 D, the energy. An analogy to the two-dimensional enstrophy does not exist, but another 3 D invariant, helicity may take on positive or negative values. On the other hand, helicity vanishes identically in 2 D.

For isotropic turbulence, the helicity must be zero, because the system is invariant under reflection. For helical flows it has been conjectured by Levich and others [8–11] that there might be an inverse cascade in 3 D, which feeds small-scale energy into large-scale coherent structures like hurricanes, cyclones, etc. For compressible fluids, Moiseev et al. [12] arrive at similar conclusions by deriving for the curl of an appropriately averaged flow field, $\boldsymbol{\Omega}$, the equation

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \tfrac{1}{2} g(0) \, \nabla \times \boldsymbol{\Omega} = \nu \, \nabla^2 \boldsymbol{\Omega} \,,$$

where $g(0)$ is a measure for the skew-symmetric part of the velocity correlation tensor. For small enough v this equation is always unstable from which the authors infer an appearance of coherent structure. Lipscombe et al. [13] consider incompressible isotropic turbulence and find that it is stable with respect to weak mean flow perturbations.

Moffatt [14] also considers helicity to be an important part of the flow and arrives at the speculation that high relative helicity should be correlated with low-energy dissipation and vice versa. Indeed, a numerical simulation of channel flow showed that velocity and vorticity vectors tend to align in regions of low dissipation [15].

In this paper we use a different approach. The general time dependent flow is expanded in eigenfields which are particularly suited to exhibit the interplay between energy and positive and negative helicity. In this way we can show that the convective term $v \cdot \nabla v$ consists of four separate interactions. Two of them display an inverse cascade, but for isotropic flows the cascading to smaller wave numbers is countered and destroyed by the other interactions. There may, however, be special initial conditions under which inverse cascades become dominant, leading to self-organization and the emergence of large structures. In the remainder of this paper we will review the two-dimensional Euler equations (Sect. II) and derive the expansion of the three-dimensional Euler equations (Section III). The resulting equations of motion are discussed in Section IV. In Sect. V two scenarios are described which may lead to the appearance of structures. The summary and conclusions in Sect. VI conclude the paper.

We consider flows described by the incompressible Navier-Stokes equations,

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 v, \quad \nabla \cdot v = 0, \qquad (1.1)$$

which are characterized by length scales and characteristic times for which the viscous term is unimportant. Hence we will omit it and work with the Euler equations

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p, \quad \nabla \cdot v = 0, \qquad (1.2)$$

which we write in the form

$$\frac{\partial v}{\partial t} - v \times (\nabla \times v) = -\nabla P, \quad P = p + \tfrac{1}{2}v^2, \quad \nabla \cdot v = 0. \ (1.3)$$

Taking the curl and introducing $\boldsymbol{\Omega} = \nabla \times v$, we obtain

$$\frac{\partial}{\partial t} \boldsymbol{\Omega} + \nabla \times (v \times \boldsymbol{\Omega}) = 0. \qquad (1.4)$$

Because the Euler equations are dissipationless, energy must be conserved if the flow is confined, i.e. $v \cdot n = 0$ on the boundary Γ where n is the normal, or v is periodic.

$$W = \tfrac{1}{2} \int_D v^2 \, \mathrm{d}\tau = \mathrm{const}. \qquad (1.5)$$

The integral extends over the volume D inside the boundary Γ. In addition there is a second integral, valid for the same constraints, the helicity

$$H = \tfrac{1}{2} \int_D v \cdot \boldsymbol{\Omega} \, \mathrm{d}\tau = \mathrm{const}. \qquad (1.6)$$

Whereas W is positive definite, the helicity may take on either sign.

II. Review of the Two-Dimensional Euler Equations

The two-dimensional Navier-Stokes equations have been studied extensively analytically [16, 17] and numerically (e.g., Seyler et al. [6]). For a review see Kraichnan and Montgomery [18]. We collect here some results which demonstrate the striking difference between two and three dimensions. Whereas the Navier-Stokes equations in configuration space (1.1) look the same in two or three dimensions, in transformed space the equation (2.7) for 2D is much simpler than the system (4.1) and (4.2) for 3D. For the latter we count eight different interaction terms on the right side under the sum whereas the former has only one interaction term. Expanding the curl operator in (1.3) and taking into account that in 2D the curl is always directed parallel to the z-axis e_z, $\boldsymbol{\Omega} = \Omega e_z$, we obtain

$$\frac{\partial}{\partial t} \Omega + v \cdot \nabla \Omega = 0. \qquad (2.1)$$

In 2D a stream function, ϕ, can be introduced,

$$v = \nabla \phi \times e_z, \qquad (2.2)$$

so that the incompressibility condition is automatically satisfied. Taking the curl, results in

$$\nabla^2 \phi = -\Omega. \qquad (2.3)$$

Introducing (2.2) in (2.1) gives the scalar equation

$$\frac{\partial}{\partial t} \Omega + (\nabla \Omega \times \nabla \phi) \cdot e_z = 0. \qquad (2.4)$$

The Eqs. (2.3) and (2.4) form a self-consistent system which conserves the two invariants, energy W and enstrophy $\mathscr{E}$,

$$W = \tfrac{1}{2} \int v^2 \, d\tau, \quad \mathscr{E} = \tfrac{1}{2} \int \Omega^2 \, d\tau, \qquad (2.5)$$

if the flow is confined or periodic, when the integrals are extended over the confined volume or the periodicity interval. Both invariants are positive definite.

We introduce periodic boundary conditions and write the vorticity as a Fourier sum

$$\Omega(r, t) = \sum_k \Omega_k(t) \exp(i k \cdot r),$$

where the (complex) coefficients Ω_k obey the equations

$$\frac{d}{dt} \Omega_k = \sum_{p+q=k} \tilde{M}_{pq-k} \, \Omega_p \, \Omega_q,$$

$$\tilde{M}_{pq-k} = \tfrac{1}{2} p \times q \cdot e_z \left(\frac{1}{p^2} - \frac{1}{q^2} \right). \qquad (2.6)$$

Introducing complex amplitudes $a_k = \Omega_k/k$, (2.6) can be written

$$\frac{d}{dt} a_k = \sum_{p+q=k} c_{p, q, -k}(p^2 - q^2) \, a_p \, a_q, \qquad (2.7)$$

$$c_{p, q, -k} = -\frac{1}{2} \frac{p \times q \cdot e_z}{pqk}, \qquad (2.8)$$

where the $c_{p, q, -k}$ are invariant with respect to a cyclic permutation of its indices.

In terms of the a_k the invariants (2.5) are given by

$$W = \tfrac{1}{2} \sum_k |a_k|^2, \quad \mathscr{E} = \tfrac{1}{2} \sum_k k^2 |a_k|^2. \qquad (2.9)$$

In (2.7) a large number of different terms, say N, act on a_k simultaneously. We may however imagine, that during a small time interval $\Delta t/N$ only one term acts, however with the N-fold intensity and all other terms are "switched off". During the next small time interval $\Delta t/N$, another term acts and so on, until all terms on the right side of (2.7) have acted. This method is known in numerical mathematics as "the method of weak approximation" or as "the method of fractional steps" or simply "splitting" [19, 20]. One can prove that the weak approximation converges to the true solution to arbitrary accuracy, if only the time interval Δt is chosen small enough. We make use of this con-

cept and consider one triplet interaction out of the many in (2.7), which conserves energy and enstrophy,

$$\dot{a}_k = c_{pq-k}(p^2 - q^2) \, a_p \, a_q,$$

$$\dot{a}_{-p} = c_{pq-k}(p^2 - k^2) \, a_q \, a_{-k}, \qquad (2.10)$$

$$\dot{a}_{-q} = c_{pq-k}(k^2 - p^2) \, a_{-k} \, a_p, \quad p + q + (-k) = 0.$$

We note that the basic interaction triplet (2.10) is invariant to time reversal if also the sign of the amplitudes are reversed. It is easily seen that

$$|a_k|^2 + |a_p|^2 + |a_q|^2 = \text{const}$$

and

$$k^2 |a_k|^2 + p^2 |a_p|^2 + q^2 |a_q^2| = \text{const}. \qquad (2.11)$$

A further integral is obtained writing

$$a_k = |a_k| \exp(i \theta_k)$$

and splitting (2.10) into real and imaginary parts. We get

$$|a_k \, a_p \, a_q| \, |\cos \theta| = \Gamma = \text{const}, \quad \theta = \theta_p + \theta_q - \theta_k. \qquad (2.12)$$

In this way (2.10) can be integrated and leads to an elliptical integral [21–24]. However, as (2.10) is valid only over a very small time, we will not write down these results. A very simple representation of the energy flow of the interaction (2.10) is obtained by introducing energy quantities $w_k = |a_k|^2$, $w_p = |a_p|^2$, $w_q = |a_q|^2$, with which (2.11) becomes

$$w_k + w_p + w_q = 2 W = \text{const},$$

$$k^2 w_k + p^2 w_p + q^2 w_q = 2 \mathscr{E} = \text{const}. \qquad (2.13)$$

These equations are mathematically identical with the equations for a spinning torque-free top [25].

In a space with w_k, w_p, w_q as Cartesian axes, the equations (2.13) represent two intersecting planes (Figure 1). Any change of energy must satisfy

$$\delta w_k + \delta w_p + \delta w_q = 0, \quad k^2 \delta w_k + p^2 \delta w_p + q^2 \delta w_q = 0,$$

or

$$\delta w \cdot e = 0, \quad \delta w \cdot s = 0, \quad \delta w = (\delta w_k, \delta w_p, \delta w_q),$$

$$e = (1, 1, 1), \quad s = (k^2, p^2, q^2).$$

The system point can only move along the direction $e \times s$, so that the energy changes become explicitly

$$\delta w_k = (q^2 - p^2) \, \dot{f}(t) \, \delta t,$$

$$\delta w_p = (k^2 - q^2) \, \dot{f}(t) \, \delta t, \qquad (2.14)$$

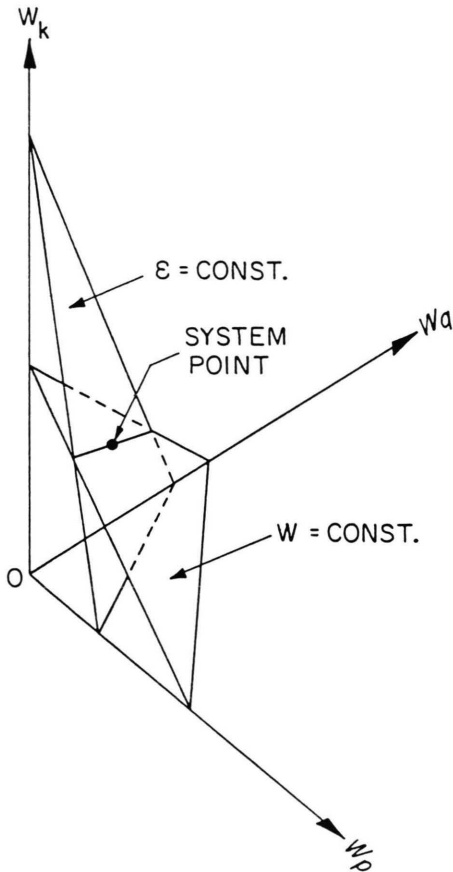$$\delta w_q = (p^2 - k^2) \, \dot{f}(t) \, \delta t.$$

Fig. 1. The trajectory of the interacting triplets (2.12) is represented in the energy space w_p, w_q, w_k. The system can move only along a straight line, the intersection of the plane $W = $ constant and $\mathscr{E} = $ constant.
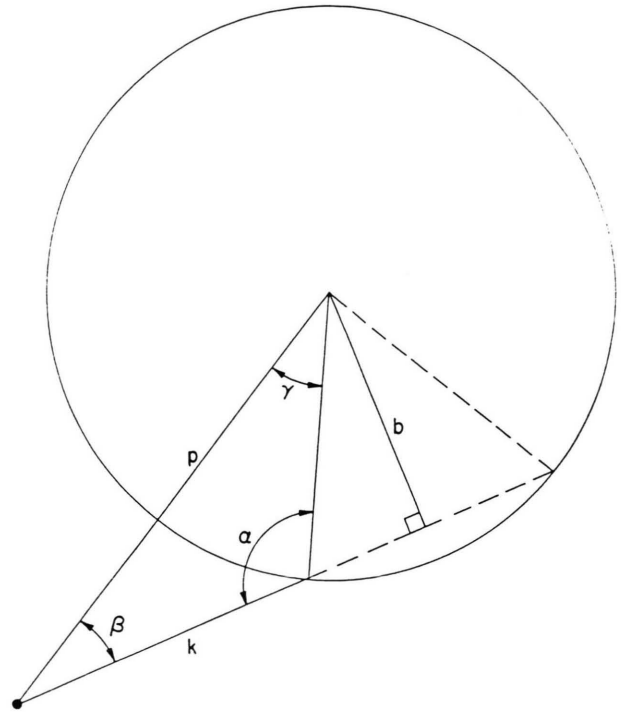

Fig. 2. The magnitude of the wave vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are held fixed, but the angle γ formed by the two vectors may vary, which in turn varies the magnitude of wave vector $\boldsymbol{k}$.

$\dot{f}(t)$ is obtained from (2.10) and is given by

$$\dot{f}(t) = -\frac{\boldsymbol{p} \times \boldsymbol{q} \cdot \boldsymbol{e}_z}{p\,q\,k} \sqrt{w_k\,w_p\,w_q}\,\cos\theta, \quad \dot{f}(t=0)=0. \quad (2.15)$$

The energy transfer from one mode to another of the triplet is proportional to $\dot{f}(t)$. We note that for certain phase relations, namely when

$$\theta = \theta_p + \theta_q - \theta_k = \pm\frac{\pi}{2} + n\pi, \quad n=1, 2, \ldots,$$

the energy transfer is blocked.

Choosing the magnitudes $|p| \neq |q|$ and keeping them fixed, but varying the angle γ (compare Fig. 2) and thus k, we note that

$$\left| \frac{\boldsymbol{p} \times \boldsymbol{q} \cdot \boldsymbol{e}_z}{p\,q\,k} \right| = \left| \frac{\sin\gamma}{k} \right| = \frac{b}{p\,q}$$

has a maximum for $\alpha = \pi/2$. The interaction term (2.8) considered as a function of $|k|$ is then greatest. On the other hand, triangles degenerating into a straight line have zero interaction strength. We may say "fat" triangles represent large, "slim" triangles weak interactions.

The square root term in (2.15) indicates that $f(t)$ is proportional to the amplitudes $|a_p\,a_q\,a_{-k}|$. Three different time scales are obtained. The fastest is between three highly excited modes. We are mainly interested in energy transfer between two highly excited modes to a third mode whose excitation level is $\mathcal{O}(\varepsilon)$. It is on this time scale that we will observe the growth of structures if the transition is unstable. If one highly excited mode transfers energy to two modes, each with excitation $\mathcal{O}(\varepsilon)$, we will obtain a time scale $\mathcal{O}(\varepsilon^2)$. Finally, if none of the modes is excited, the process is unimportant and can be neglected.

It should be stressed that the magnitude of $\dot{f}(t)$ characterizes the speed with which energy is transferred. Numerical solutions of the triplet (2.10) show this very clearly: whenever one or two amplitudes are
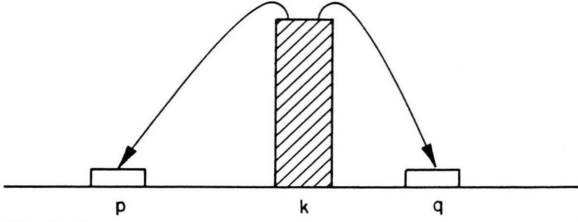
Fig. 3. An energy transfer from k to q is only possible if simultaneously energy is also transferred from k to p, where $p < k < q$.

zero, the rate of energy transfer is zero. The time scale defined by $\dot{f}(t)$ may be quite different from another characteristic time, let us call it the amplitude time scale, which characterizes the growth of one mode. In order to obtain the amplitude time scale we differentiate the third of the Eqs. (2.10) and insert the first two. The result is

$$\ddot{a}_{-p} = |c|^2 (q^2 - k^2)[(k^2 - p^2)|a_k|^2 + (p^2 - q^2)|a_q|^2] a_{-p}$$
$$= \gamma^2 a_{-p}.$$

γ is the instantaneous amplitude time scale of a_{-p}.

For $p < k < q$, $\gamma^2 > 0$, let $|a_k|^2 = \mathcal{O}(1)$, $|a_p|^2 = \mathcal{O}(\varepsilon)$, and $|a_q|^2 = \mathcal{O}(\varepsilon)$, where $\varepsilon \ll 1$. We find

$$\gamma = |c|\sqrt{(q^2 - k^2)(k^2 - p^2)|a_k|^2} > 0.$$

If $|a_q|^2 = \mathcal{O}(1)$, we have

$$\gamma = |c|\sqrt{(q^2 - k^2)(k^2 - p^2)|a_k|^2 - (q^2 - p^2)|a_q|^2}.$$

In the first case γ is always > 0; the system is unstable and the mode p grows exponentially. In the second case the radicant may become negative if $|a_q|^2$ is large enough and then the system performs small-amplitude oscillations around the equilibrium state. The system is stable. $|\gamma|$ may however be large in both cases.

The energy and enstrophy conservation is contained in the factors $(q^2 - p^2)$, etc. in (2.14) and is shown in Figure 3. If energy cascades from k to the larger q, some energy also has to flow to the smaller wavelength p. Because of the time reversibility, energy can also flow from p and q into k. However, there are more q-states than p-states, because $q > p$. On the statistical average energy tends to occupy the states with larger wave numbers where eventually the energy is dissipated by viscosity. Concomitant with this "regular" flow of energy is the energy flow to lower wave numbers. We define this process to be an inverse cascade. It was first discovered by Fjørtoft already in 1953. Seyler et al. [6] demonstrated for a system, trun-

cated at a maximum wave number, that under certain initial conditions, the inverse cascade may result in a concentration of energy at the lowest possible wave number. Numerical simulations show the development of two big vortices of opposite sign giving clear indication of the self-organization of a two dimensional flow. However, Seyler et al. also showed that this must not always be the case but depends on the initial conditions. Using methods of statistical mechanics, the case of strong self-organization was shown to possess a "negative temperature". A partition function can be derived, given by

$$Z = \prod_{\nu=1}^{N} Z_\nu, \quad Z_\nu = (\beta + \alpha k_\nu^2)^{-1}, \tag{2.16}$$

from which follows the energy

$$W = -\frac{\partial}{\partial \beta} \ln Z = \sum_{\nu=1}^{N} \frac{1}{\beta + \alpha k_\nu^2}, \tag{2.17}$$

and the enstrophy

$$\mathscr{E} = -\frac{\partial}{\partial \alpha} \ln Z = \sum_{\nu=1}^{N} \frac{k_\nu^2}{\beta + \alpha k_\nu^2}. \tag{2.18}$$

We order the available k-values, so that

$$k_0^2 \leqq k_1^2 \leqq k_2^2 \leqq \ldots. \tag{2.19}$$

The ratio of enstrophy to energy is then given by

$$\frac{\mathscr{E}}{W} = k_0^2 \frac{1 + \dfrac{k_1^2}{k_0^2}\dfrac{\beta + \alpha k_0^2}{\beta + \alpha k_1^2} + \dfrac{k_2^2}{k_0^2}\dfrac{\beta + \alpha k_0^2}{\beta + \alpha k_2^2} + \cdots + \dfrac{k_N^2}{k_0^2}\dfrac{\beta + \alpha k_0^2}{\beta + \alpha k_N^2}}{1 + \dfrac{\beta + \alpha k_0^2}{\beta + \alpha k_1^2} + \dfrac{\beta + \alpha k_0^2}{\beta + \alpha k_2^2} + \cdots + \dfrac{\beta + \alpha k_0^2}{\beta + \alpha k_N^2}}. \tag{2.20}$$

With (2.19) and (2.20) one can convince oneself that $\mathscr{E}/W$ is a monotonically increasing function of β. For $\beta = 0$, (2.20) becomes

$$\left\{ \frac{1}{N} \sum_{\nu=1}^{N} \frac{1}{k_\nu^2} \right\}^{-1} = [\langle k^{-2} \rangle]^{-1}.$$

We have a negative temperature case if

$$\frac{\mathscr{E}}{W} < [\langle k^{-2} \rangle]^{-1}. \tag{2.21}$$

The inequality (2.21) is a criterion which does no longer have any reference to statistical mechanics. We expect the development of structures if condition (2.21) is satisfied and the structures can be seen the more distinctly the larger the margin by which inequality (2.21) is satisfied.

III. Expansion of the Three-Dimensional Euler Equations

In an expansion which is often used (compare, e.g., Leslie [26]), one uses a Fourier representation of the incompressible flow

$$v(r) = \sum_k v(k) \exp(i\,k \cdot r), \quad v(k) \cdot k = 0$$

and inserts it into (1.2). One obtains equations for the components $v(k)$,

$$\dot{v}(k) = -i\,k \cdot \mathbf{V} \cdot (\mathbf{I} - \hat{k}\,\hat{k}), \tag{3.1}$$

where the symmetric tensor $\mathbf{V}$ is given by

$$\mathbf{V} = \tfrac{1}{2} \sum_{p+q=k} (v(p)\,v(q) + v(q)\,v(p)). \tag{3.2}$$

Using index notation for Cartesian coordinates, with $v_i(q)$ being the i-th component of the Fourier component of the mode q, (3.1) becomes

$$\dot{v}_\nu(k) = M_{\nu j i}\,V_{ij}(k) \tag{3.3}$$

with

$$V_{ij}(k) = \tfrac{1}{2} \sum_{p+q=k} [v_i(p)\,v_j(q) + v_j(q)\,v_i(p)] \tag{3.4}$$

and

$$M_{\nu j i} = -\frac{i}{2}\left\{ k_i\left(\delta_{j\nu} - \frac{k_j k_\nu}{k^2}\right) + k_j\left(\delta_{i\nu} - \frac{k_\nu k_i}{k^2}\right)\right\}. \tag{3.5}$$

This way of writing the Euler equations is concise, but it is difficult to interpret them physically. In particular, (3.3) to (3.5) do not exhibit clearly the conservation of energy and helicity.

In quantum mechanical perturbation theory, one often spans the Hilbert space by eigenfunctions of the stationary, unperturbed Hamiltonian because this representation is particularly well adapted to a number of problems. Even though the Euler equations are very different and in addition nonlinear, one might think of spanning the Hilbert space of incompressible flows in terms of stationary solutions of the Euler equations. Equation (1.4) shows that if

$$\boldsymbol{\Omega} = \boldsymbol{V} \times \boldsymbol{v} \equiv L\boldsymbol{v} = \lambda\boldsymbol{v}, \tag{3.6}$$

$\partial\boldsymbol{\Omega}/\partial t = 0$ or $\boldsymbol{V} \times \boldsymbol{v} = \boldsymbol{\Omega} = $ constant in time. Together with $\boldsymbol{V} \cdot \boldsymbol{v} = 0$ and boundary conditions, the velocity field can be constructed which is also independent of time. Therefore, solutions of (3.6) offer themselves naturally to span a Hilbert space for all incompressible flows, if the set of solutions is complete. In general λ in (3.6) may still depend on spatial coordinates. However, if λ is constant, we have, again in analogy to quantum mechanics, an eigenvalue problem.

The helical decomposition has been first introduced by Lesieur in 1972 and is described in Lesieur [27]. Helical turbulence has been studied, among others, by Kraichnan [28] and André and Lesieur [29]. We define an inner product of two vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ by

$$\langle v, u \rangle = \int v^*(r) \cdot u(r)\, \mathrm{d}^3 r. \tag{3.7}$$

Then it is easy to see that the curl is a formally self-adjoint operator, and a self-adjoint operator if

$$\int_\Gamma v^* \times u \cdot n\, \mathrm{d}\sigma = 0, \tag{3.8}$$

where the integral extends over the boundary of the simply connected volume and n is the normal on the boundary.

We find

$$\begin{aligned}
\langle v, Lu \rangle &= \int v^* \cdot V \times u\, \mathrm{d}^3 r \\
&= \int V \cdot (u \times v^*)\, \mathrm{d}^3 r + \int (V \times v)^* \cdot u\, \mathrm{d}^3 r \\
&= \langle Lv, u \rangle - \int v^* \times u \cdot n\, \mathrm{d}\sigma,
\end{aligned}$$

which shows that the curl is formally self-adjoint. The surface integral (3.8) vanishes for periodic flows and the integrand vanishes identically if at the boundary unique tangential directions are prescribed for all flows. Under these conditions the curl is self-adjoint. The flow can then continuously be connected to other fields, which may be of a different type, for example, non-helical or laminar (compare, e.g., Lüst and Schlüter [30]).

From the self-adjointness of the operator L in (3.6) follows immediately that all eigenvalues λ are real and that the eigenfunctions for different λ are orthogonal to each other. Further, we note that (3.6) is invariant to the substitution

$$v \rightarrow -r, \quad \lambda \rightarrow -\lambda.$$

Eigenvalues appear always in pairs, a positive and a negative one, and to these are associated two eigen-flows $v_\lambda^+(r)$ and $v_\lambda^-(r)$ such that

$$V \times v_\lambda^\pm(r) = \pm |\lambda|\, v_\lambda^\pm(r), \tag{3.9}$$

where the $v_\lambda^\pm$ are orthonormal so that

$$\langle v_\lambda^\pm, v_\mu^\pm \rangle = \delta_{\lambda\mu} \quad \text{and} \quad \langle v_\lambda^\pm, v_\lambda^\mp \rangle = 0. \tag{3.10}$$

The geometrical significance is evident: to each helical flow described by a right-handed screw there corresponds a mirrored helical flow described by a left-handed screw.

It is well possible that eigenflows are degenerate, i.e., two or more different flows may correspond to the same eigenvalue. As is well kown, the flows can be chosen in such a way that they are orthogonal. Taking the scalar product of (3.9) with $[v_\lambda^\pm(r)]^*$ and integrating over the volume results in

$$\frac{H}{W} = \pm |\lambda|.$$

A solution of the Euler equations described by one of the eigenflows (3.9) has a helicity over energy ratio given by the sign and magnitude of its eigenvalue.

Taking the curl of (3.9) and using the incompressibility condition results in the vector Helmholtz equation [31].

$$\nabla^2 v + \lambda^2 v = 0. \tag{3.11}$$

In solving (3.11), care has to be taken that v is divergence free. This is accomplished automatically by defining a vector potential A by $V \times A = v$ and letting A satisfy (3.11). Functions, which are solutions of (3.9) or (3.11) have been used for various purposes in the literature [30, 32–36].

For our purposes it is sufficient to consider the simplest solutions which are closely related to conventional Fourier modes. We take a large cubic box of side length L and impose periodic boundary conditions on a flow described by

$$v = \sum_K (a_k^+ u_k^+ + a_k^- u_k^-) \exp(i k \cdot r). \tag{3.12}$$

Reality of the flow requires

$$(u_{-k}^\pm)^* = u_k^\pm,$$
$$(a_{-k}^\pm)^* = a_k^\pm.$$

$u_k^\pm \exp(i k \cdot r)$ satisfies (3.6), from which follows

$$i k \times u_k^\pm = \pm \lambda_k u_k^\pm. \tag{3.13}$$

Here and in the following the subscripts k, p, and q denote wave vectors rather than spatial indices. It is clear that only a complex vector can satisfy the equation. We find as a solution

$$u_k^\pm = \frac{1}{\sqrt{2}} (\hat{b}_k \pm i \hat{k} \times \hat{b}_k), \tag{3.14}$$

where $\hat{k}$ is the unit vector in the k-direction and $\hat{b}_k$ is an arbitrary unit vector which satisfies the incompressibility condition

$$k \cdot \hat{b}_k = 0. \tag{3.15}$$

From the choice of orthogonal eigenflows follows

$$u_k^\pm \cdot (u_k^\pm)^* = 1, \quad u_k^\pm \cdot (u_k^\mp)^* = 0. \tag{3.16}$$

With

$$\hat{b}_k = \hat{b}_{-k}, \tag{3.17}$$

the reality conditions are satisfied.

The solution (3.13) is determined only up to a complex phase factor $\exp(i \phi_k)$, where ϕ_k can arbitrarily be chosen.

From (3.13) and (3.14) we find

$$\lambda_k = |k|. \tag{3.18}$$

In order to obtain a better understanding of the flow represented by (3.14) we let k point along the z-axis of a Cartesian coordinate system and put $\hat{b}_k = e_x$.

Then (3.18) becomes

$$v_k^\pm = \frac{1}{\sqrt{2}} (e_x \pm i e_y).$$

With $a_k^\pm = \sqrt{2} A$ this corresponds to the real flow

$$v(x, y, z) = A (\cos kz, \mp \sin kz, 0). \tag{3.19}$$

The flow lines are straight lines in the plane $z = z_0$ and as z_0 increases, the direction of the flow lines rotates (compare Figure 4).

Adding three orthogonal flows (3.19) together by permuting the coordinate axes, we obtain

$$v(x, y, z) = (A \cos kz + C \sin ky,$$
$$B \cos kx + A \sin kz, C \cos ky + B \sin kz).$$

This is the so-called ABC flow which has been recently the subject of an extensive investigation [37].

Our representation of eigenmodes of a hydrodynamic flow corresponds to the representation of circularly polarized waves in optics [38], whereas the standard Fourier representation in hydrodynamics corresponds to linearly polarized light waves. Superposition of right and left circularly polarized modes produces linearly polarized modes in optics and hydrodynamics. For the latter the helicity is then zero. The analogy ends if we consider interactions of modes. In optics the interactions are essentially linear, in hydrodynamics essentially nonlinear. There is – at least in linear optics – no analogy of the mixing and cascading of modes which we observe in hydrodynamics.

It is easy to demonstrate the completeness of the eigenvectors (3.14). The most general divergence free
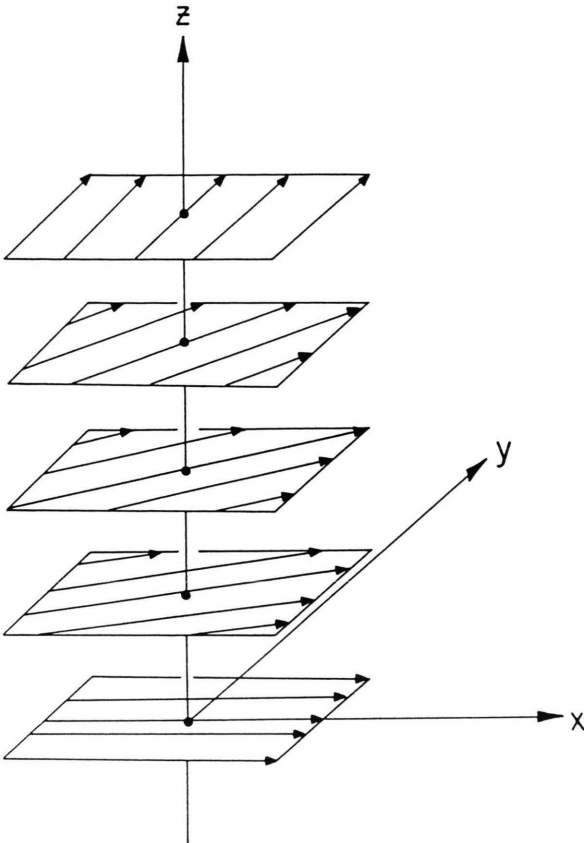
Fig. 4. The simplest helical flow as described in (3.21). The flow satisfies $V \times v = \lambda v$.

Fig. 5. An example of energy transfer which conserves energy and helicity without any "inverse" cascade.

vector field can be written in the form

$$v = \frac{1}{\sqrt{2}} \sum_k (\xi_k \hat{\boldsymbol{b}}_k + \eta_k \hat{\boldsymbol{k}} \times \hat{\boldsymbol{b}}_k) \exp(i\boldsymbol{k} \cdot \boldsymbol{r}). \quad (3.20)$$

ξ_k and η_k are arbitrary complex numbers and $\hat{\boldsymbol{b}}_k$ satisfies (3.15). Substitution of

$$\xi_k = a_k^+ + a_k^-, \quad \eta_k = -\frac{1}{i}(a_k^+ - a_k^-)$$

and use of (3.15) leads directly to (3.14).

The null space of the curl operator in (3.6) contains all flows with vanishing vorticity which are consistent with the boundary conditions. From $V \times v = 0$ follows $v = -\nabla\phi$, and the potential ϕ satisfies $\nabla^2\phi = 0$. These flows are potential flows and satisfy the Laplace equation.

Periodic flow in a cube enforces the form

$$\psi = \boldsymbol{r} \cdot \mathbf{V}_0 + \sum_k \psi_k \exp(i\boldsymbol{k} \cdot \boldsymbol{r}), \quad \psi_k = \text{const}.$$

The Laplace equation gives

$$k^2 \psi_k = 0.$$

The condition $\psi_k = 0$ describes a homogeneous flow, which can be removed by a Galilei transformation. We conclude that for nonsingular flows and periodic boundary conditions, the null space of the curl contain only the null vector.

The invariants energy and helicity (1.4) and (1.5) can now be expressed by the coefficients $a_k^\pm$, using (3.12), (3.7) and (3.18):

$$W = \frac{1}{2} \sum_k (|a_k^+|^2 + |a_k^-|^2),$$

$$H = \frac{1}{2} \sum_k k(|a_k^+|^2 - |a_k^-|^2). \quad (3.21)$$

By expansion of the vector field v in eigenfunctions of the curl, each a_k^+ or a_k^- is uniquely associated with a specific sign of the helicity. However, contrary to the two-dimensional case the conservation laws do by no means enforce an inverse cascade. Let, for example, energy be initially centered around wave number k with positive helicity as in Figure 5. Assume that after some time one-half of the energy has been transformed to energy with negative helicity, centered around p and the other half has been shifted to wave number q. Energy has been trivially conserved and helicity conservation gives $q - p \approx 2k$. No energy has migrated to larger structures. The global conservation theorems do not favor per se self-organization.

IV. The Equations of Motion

We insert ansatz (3.12) into the Euler equations, select all terms with $\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$ and get

$$\dot{a}_k^+\,\boldsymbol{v}_k^+ + \dot{a}_k^-\,\boldsymbol{v}_k^- - \sum_{p+q=k}\{q\,a_p^+\,a_q^+\,\boldsymbol{v}_p^+\times\boldsymbol{v}_q^+ + q\,a_p^-\,a_q^+\,\boldsymbol{v}_p^-\times\boldsymbol{v}_q^+$$
$$-q\,a_p^+\,a_q^-\,\boldsymbol{v}_p^+\times\boldsymbol{v}_q^- - q\,a_p^-\,a_q^-\,\boldsymbol{v}_p^-\times\boldsymbol{v}_q^-\} = -i\boldsymbol{k}\,P_k.$$

The sum extends over all vectors $\boldsymbol{q}$ and $\boldsymbol{p}$ for which $\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}$. $\boldsymbol{p}$ and $\boldsymbol{q}$ being dummy indices, we add the sum with $\boldsymbol{p}$ and $\boldsymbol{q}$ interchanged, correcting the duplication by a factor 1/2. Also, we take the inner product with $\boldsymbol{v}_k^+$ and $\boldsymbol{v}_k^-$, respectively, taking into account their orthonormality. This procedure results in the equation of motion for $\dot{a}_k^\pm$:

$$\dot{a}_k^+ = \tfrac{1}{2}\sum_{p+q=k}\{(p-q)[-a_p^+\,a_q^+\,M_{pq-k}^{+++} + a_p^-\,a_q^-\,M_{pq-k}^{--+}]$$
$$+ (p+q)[-a_p^+\,a_q^-\,M_{pq-k}^{+-+} + a_p^-\,a_q^+\,M_{pq-k}^{-++}]\},\quad (4.1)$$

$$\dot{a}_k^- = \tfrac{1}{2}\sum_{p+q=k}\{(p-q)[-a_p^-\,a_q^-\,M_{pq-k}^{---} + a_p^+\,a_q^+\,M_{pq+k}^{++-}]$$
$$+ (p+q)[-a_p^-\,a_q^+\,M_{pq-k}^{-+-} + a_p^+\,a_q^-\,M_{pq-k}^{+--}]\},\quad (4.2)$$

The interaction coefficients M are scalar triple products of three eigenvectors (3.14),

$$M_{pq-k}^{\mathcal{E}_p\,\mathcal{E}_q\,\mathcal{E}_{-k}} = \boldsymbol{v}_p^{\mathcal{E}_p}\cdot(\boldsymbol{v}_q^{\mathcal{E}_q}\times\boldsymbol{v}_{-k}^{\mathcal{E}_{-k}})\qquad (4.3)$$

$$\text{with}\quad \mathcal{E}_i = \pm 1,\quad i = p, q, k.$$

All combinations of $(+)$ and $(-)$ occur in the upper indices of the M. The M are invariant with respect to cyclic permutations of the upper and lower indices

$$M_{pq-k}^{\mathcal{E}_p\,\mathcal{E}_q\,\mathcal{E}_{-k}} = M_{q-kp}^{\mathcal{E}_q\,\mathcal{E}_{-k}\,\mathcal{E}_p} = M_{-kpq}^{\mathcal{E}_{-k}\,\mathcal{E}_p\,\mathcal{E}_q}\qquad (4.4)$$

but change sign if two pairs of upper and lower indices are interchanged:

$$(4.5)$$

$$M_{pq-k}^{\mathcal{E}_p\,\mathcal{E}_q\,\mathcal{E}_{-k}} = -M_{qp-k}^{\mathcal{E}_q\,\mathcal{E}_p\,\mathcal{E}_{-k}} = -M_{-kqp}^{\mathcal{E}_{-k}\,\mathcal{E}_q\,\mathcal{E}_p} = -M_{p-kq}^{\mathcal{E}_p\,\mathcal{E}_{-k}\,\mathcal{E}_q}.$$

The interaction constants are in general complex because the vectors $\boldsymbol{v}_k^\pm$ are complex and undetermined up to a phase factor, which, if necessary, can be fixed by an appropriate definition. We are therefore only interested in the moduli of the M, which are calculated in the appendix and given by (compare Figure 6)

$$|M_{pq-k}^{\mathcal{E}_p\,\mathcal{E}_q\,\mathcal{E}_{-k}}| = |\mathcal{E}_p\sin\alpha_p + \mathcal{E}_q\sin\alpha_q + \mathcal{E}_{-k}\sin\alpha_k|.\quad (4.6)$$

With the lower indices held fixed, all M with mixed upper indices are smaller than those with only plus or minus signs, e.g.,

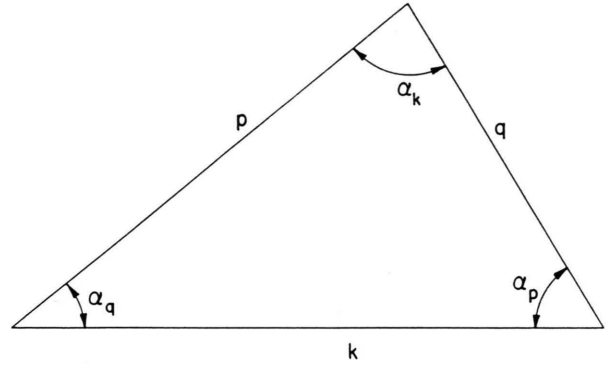$$|M_{pq-k}^{++-}| < |M_{pq-k}^{+++}| = |M_{pq-k}^{---}|.\qquad (4.7)$$

Fig. 6. Definition of quantities entering the interaction term $|M_{pq-k}^{\mathcal{E}_p\,\mathcal{E}_q\,\mathcal{E}_{-k}}| = |\mathcal{E}_p\sin\alpha_p + \mathcal{E}_q\sin\alpha_q + \mathcal{E}_{-k}\sin\alpha_k|$.

In order to interpret (4.1) and (4.2) we imagine again that all interactions except for one triplet are turned off during a very small time Δt, then another triplet is turned on, etc. We realize that we have four basic different interactions. They are:

1. Interaction I^+; three positive helicity modes interact:

$$I^+:\quad 2\dot{a}_k^+ = -(p-q)\,a_p^+\,a_q^+\,M_{pq-k}^{+++},$$
$$2\dot{a}_{-p}^+ = -(q-k)\,a_q^+\,a_{-k}^+\,M_{q-kp}^{+++},$$
$$2\dot{a}_{-q}^+ = -(k-p)\,a_{-k}^+\,a_p^+\,M_{-kpq}^{+++}.\qquad (4.8)$$

They satisfy the invariants

$$|a_k^+|^2 + |a_p^+|^2 + |a_q^+|^2 = 2W = \text{const},$$
$$k\,|a_k^+|^2 + p\,|a_p^+|^2 + q\,|a_q^+|^2 = 2H = \text{const}.\quad (4.9)$$

2. Interaction I^-; three negative helicity modes interact. This interaction is the mirror image of I^+.

$$I^-:\quad 2\dot{a}_k^- = (p-q)\,a_p^-\,a_q^-\,M_{pq-k}^{---},$$
$$2\dot{a}_{-p}^- = (q-k)\,a_q^-\,a_{-k}^-\,M_{q-kp}^{---},$$
$$2\dot{a}_{-q}^- = (k-p)\,a_{-k}^-\,a_p^-\,M_{-kpq}^{---},\qquad (4.10)$$

with

$$|a_k^-|^2 + |a_p^-|^2 + |a_q^-|^2 = 2W = \text{const},$$
$$-k\,|a_k^-|^2 - p\,|a_p^-|^2 - q\,|a_q^-|^2 = 2H = \text{const}.\quad (4.11)$$

3. Interaction II^+; two positive and one negative helicity modes interact.

$$II^+:\quad 2\dot{a}_k^+ = -(p+q)\,a_p^+\,a_q^-\,M_{pq-k}^{+-+},$$
$$2\dot{a}_{-p}^+ = (q+k)\,a_q^-\,a_{-k}^+\,M_{q-kp}^{-++},$$
$$2\dot{a}_{-q}^+ = -(k-p)\,a_{-k}^+\,a_p^+\,M_{-kpq}^{++-}.\qquad (4.12)$$

with

$$|a_k^+|^2 + |a_p^+|^2 + |a_q^-|^2 = 2W = \text{const},$$

$$k|a_k^+|^2 + p|a_p^+|^2 - q|a_q^-|^2 = 2H = \text{const}. \quad (4.13)$$

4. Interaction II$^-$; two negative and one positive helical mode interact. This interaction is the mirror image of interaction II$^+$.

$$\text{II}^-: \quad 2\dot{a}_k^+ = (p-q)\, a_p^-\, a_q^-\, M_{pq-k}^{--+},$$

$$2\dot{a}_{-p}^- = (q+k)\, a_q^-\, a_{-k}^+\, M_{q-kp}^{-+-},$$

$$2\dot{a}_{-q}^- = -(k+p)\, a_k^+\, a_p^-\, M_{-kpq}^{+--}, \quad (4.14)$$

with

$$|a_k^+|^2 + |a_p^-|^2 + |a_q^-|^2 = 2W = \text{const},$$

$$k|a_k^+|^2 - p|a_p^-|^2 - q|a_q^-|^2 = 2H = \text{const}. \quad (4.15)$$

The relations (4.8) to (4.15) are displayed graphically in Figure 7. In the first column is listed the name given to the interactions. There are four physically different interactions I$^+$, I$^-$, II$^+$, II$^-$. In the second column the pertinent energy equations for the triplet are written down as well as the expressions for the energy and helicity. The third column contains the graphical representation of these equations and the associated flow of the energy. The triplet equations are time reversible. Thus the arrows of the energy flow in the figures may as well point in the opposite direction. The fourth column finally contains remarks on noteworthy properties of the interactions.

It is now not difficult to prove constancy of energy and helicity from (4.1) and (4.2). Multiply (4.1) with a_{-k}^+, (4.2) with a_{-k}^-, add the equations, sum over k and add the conjugate complex. The resulting sum on the right-hand side decomposes into annihilating triplets and vanishes, so that we get the first equation (3.21). Corresponding arguments hold for the helicity, for which we obtain the second equation (3.21).

We discuss now the basic interactions I$^\pm$ and II$^\pm$. They are time reversible, so we may confine ourselves to consider transitions with one donor and two receiver modes, without limiting the generality of the argument. Using the same methods as in the two-dimensional case, we derive for the energy transfer for I$^+$:

$$\dot{w}_k^+ = -(p-q)\, \dot{f}(t),$$

$$\dot{w}_p^+ = -(q-k)\, \dot{f}(t),$$

$$\dot{w}_q^+ = -(k-p)\, \dot{f}(t), \quad (4.16)$$

$$\dot{f}(t)\sqrt{w_k^+\, w_p^+\, w_q^+}\,|M|\cos\theta, \quad \theta = \theta_p + \theta_q - \theta_k.$$

Let us assume $\dot{w}_k^+ < 0$, $\dot{w}_p^+$, $\dot{w}_q^+ > 0$. Then follows for $\dot{f}(t) > 0$ from (4.16): $p-q > 0$, $q-k < 0$, $k-q < 0$, or $q < k < p$. For $\dot{f} < 0$ follows $p < k < q$. In other words, the donor mode (if there is only one) is always in between the two receptor modes as shown in Figure 7. As an example take $q = 3$, $k = 4$, $p = 5$. Then the energy transfer rate to q is $|\dot{w}_q^+/\dot{w}_k^+| = 1/2$ and to p is $|\dot{w}_p^+/\dot{w}_k^+| = 1/2$. Consider a distribution of energy along the one-dimensional wave number axis. Then the "center of energy" is left invariant by triplet interactions of the form (4.16).

The interactions I$^\pm$ represent an inverse cascade: if energy flows to a larger wave number, some energy must also flow to a shorter wave number in order to conserve helicity. This interaction is similar to the (only) interaction term in two dimensions. However, in 2D we find the square of the wave numbers as coefficients, whereas in 3D it is only the modulus of the wave number.

We also note that the interactions I$^\pm$ take place only between modes with the same sign of the helicity. There is no "mixing" of positive and negative helicity. If most of the energy is trapped in the smallest possible mode of the system, the interactions I$^+$ are unable to remove it from there.

The interactions II$^+$ are characteristic for three-dimensional flow. There are no analogous terms in 2D. The significant difference to the interactions I$^\pm$ is that in the helicity conservation for a triplet the factors k, p, q, do not have the same sign (it is equal to the sign of the helicity index of the respective energy).

Interaction II$^+$ is characterized by the energy transfer equations

$$\dot{w}_k^+ = -(p+q)\, \dot{f}(t),$$

$$\dot{w}_p^+ = (q+k)\, \dot{f}(t),$$

$$\dot{w}_q^- = -(k-p)\, \dot{f}(t), \quad (4.17)$$

$$\dot{f}(t) = \sqrt{w_k^+\, w_p^+\, w_q^-}\,|M|\cos\theta, \quad \theta = \theta_p + \theta_q - \theta_k.$$

Assuming $\dot{w}_k^+ < 0$, $\dot{w}_p^+$, $\dot{w}_q^- > 0$, we find $\dot{f}(t) > 0$ and obtain the condition $k < p$, and q satisfies $\mathbf{q} = -\mathbf{p} + \mathbf{k}$. If $q > k$ as shown in Fig. 7, all energy cascades to higher wave numbers. As an example put $k = 4$, $p = 5$, $q = 6$, and let k be the donor mode. Then the p and q modes receive the following fractions of the energy: $|\dot{w}_p^-/\dot{w}_k^+| = 10/11$ and $|\dot{w}_q^+/\dot{w}_k^+| = 1/11$.

In the example $p > k$. However q may also be chosen smaller than k. Then this interaction lets energy again flow to smaller wave numbers. Let $k = 4$, $p = 5$,
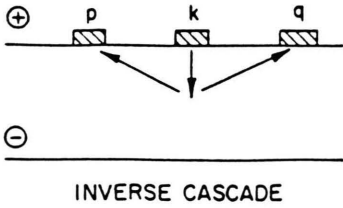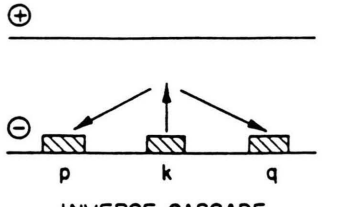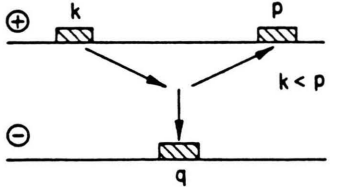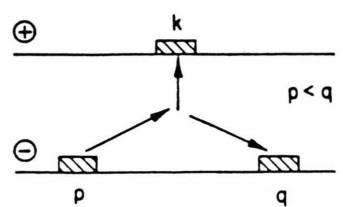
TYPE	EQUATIONS	GRAPH	REMARKS
I^+	$\dot{w}_k^+ = -(p-q)f(t)$ $\dot{w}_p^+ = -(q-k)f(t)$ $\dot{w}_q^+ = -(k-p)f(t)$	$\oplus$ p k q $\ominus$ INVERSE CASCADE	The middle mode is always sole donor or sole receptor.
I^-	$\dot{w}_k^- = -(p-q)f(t)$ $\dot{w}_p^+ = (q-k)f(t)$ $\dot{w}_q^+ = (k-p)f(t)$	$\oplus$ $\ominus$ p k q INVERSE CASCADE	The middle mode is always sole donor or sole receptor.
II^+	$\dot{w}_k^+ = -(p+q)f(t)$ $\dot{w}_p^+ = (q+k)f(t)$ $\dot{w}_q^- = -(k-p)f(t)$	$\oplus$ k p $\ominus$ k < p q REGULAR OR INVERSE CASCADE	The magnitude of q is restricted by $q = |\mathbf{k} - \mathbf{p}|$.
II^-	$\dot{w}_k^+ = (p-q)f(t)$ $\dot{w}_p^- = (q+k)f(t)$ $\dot{w}_q^- = -(k+p)f(t)$	$\oplus$ k $\ominus$ p < q p q REGULAR OR INVERSE CASCADE	The magnitude of k is restricted by $k = |\mathbf{p} + \mathbf{q}|$.

Fig. 7. Graphical display of the four different interactions of the Euler equations, which have been brought into the form (4.1) and (4.2).

$q = 3$. k is again the only donor mode and we get $|\dot{w}_p^- / \dot{w}_k^+| = 7/8$ and $|\dot{w}_q^+ / \dot{w}_k^+| = 1/8$, i.e., 88% of the transferred energy flows to larger wavelengths. (4.17) indicates that one of the positive helical modes is always donor, the other receptor. It is thus not possible that the negative helicity wave is the only donor, and all the possible transitions are covered in the graph of Fig. 7 with the proviso that $q = |\mathbf{p} + \mathbf{k}|$ and the arrows may be reversed.

We note that the moduli of the type II coupling constants M are in general smaller than the ones of type I becaue the $\mathscr{E}$'s in (4.6) have not all the same sign. Type II interactions bring about a mixing of positive and negative helicity. If initially there was only, say
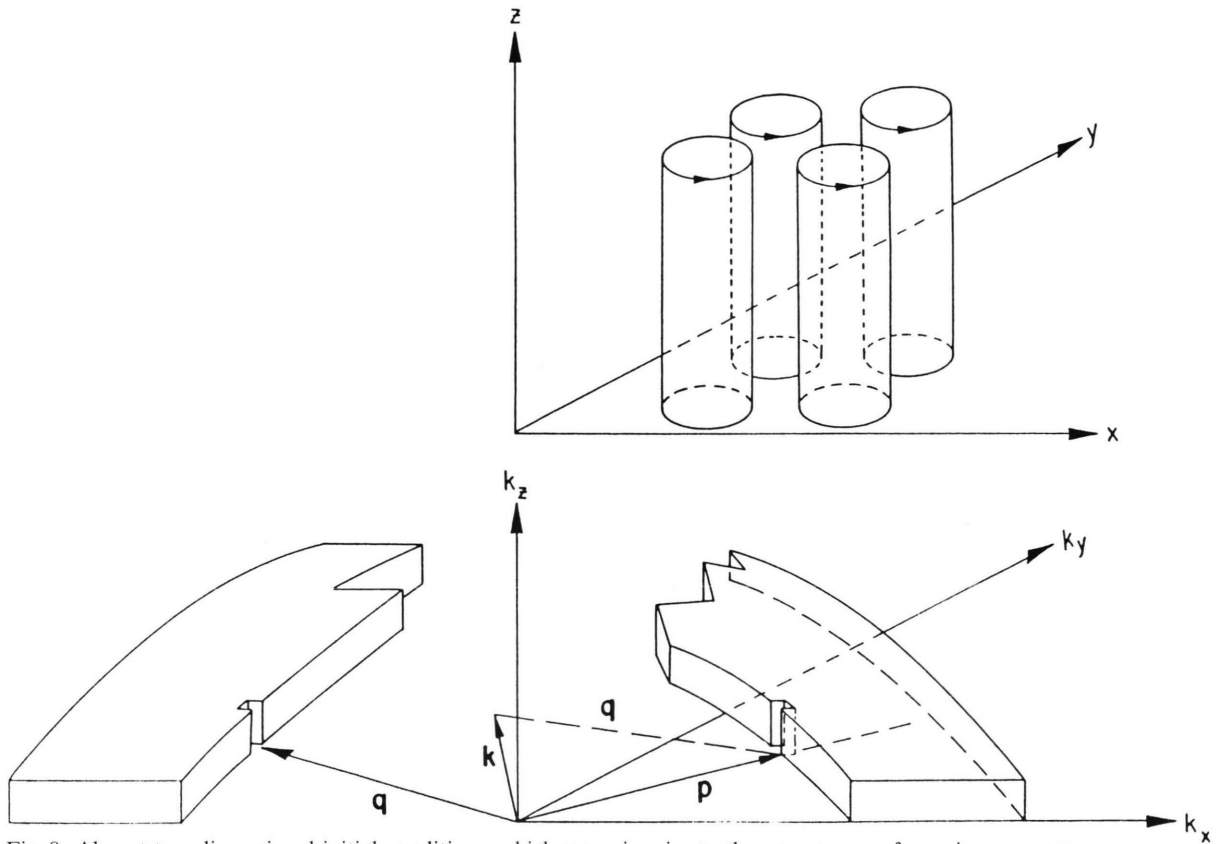
Fig. 8. Almost two-dimensional initial conditions, which may give rise to the appearance of growing structures.

positive helicity, then, in the course of time, negative helicity will be generated in such a way that the total helicity will be conserved.

II^- is again the mirror image of II^+ and our remarks apply mutatis mutandis to that interaction.

V. Two Hydrodynamic Scenarios

In this paragraph we would like to speculate in a nonrigorous way, how self-organization can take place in a flow. We consider an initial value problem, assuming that during the time of observation the action of the viscous terms are negligible, at least for those modes which are important for self-organization and whose wave vectors are sufficiently small.

The first case is a configuration which is almost two dimensional. "Almost" means here that the existing structures have very little variation, along the z-axis,

say. Consequently, the distribution of excited modes in k-space is flat, like a pancake, with the k_z-axis being the normal on the pancake, as in Figure 8. The vorticity is almost perpendicular to the flow, so $H \approx 0$ and positive and negative helicity is evenly distributed.

Let the pancake distribution have a hole in the middle so that it looks like a slice of canned pineapple. We look into triplet interactions which have an energy time scale $\mathcal{O}(\varepsilon)$. For this to be true only one amplitude may be of order ε in a triplet interaction whereas the two others must have a substantial excitation level. However as these mode vectors lie almost in a plane, the z-component of the third vector must also be small, assuming a "fat" triangle of interacting vectors. It follows that initially the interactions are also almost two dimensional. Under these circumstances we have an inverse cascade which tends to fill in the hole in the pineapple slice. In configuration space this corresponds to the growth of structures. Eventually the modes with larger components along the k_z-axis will
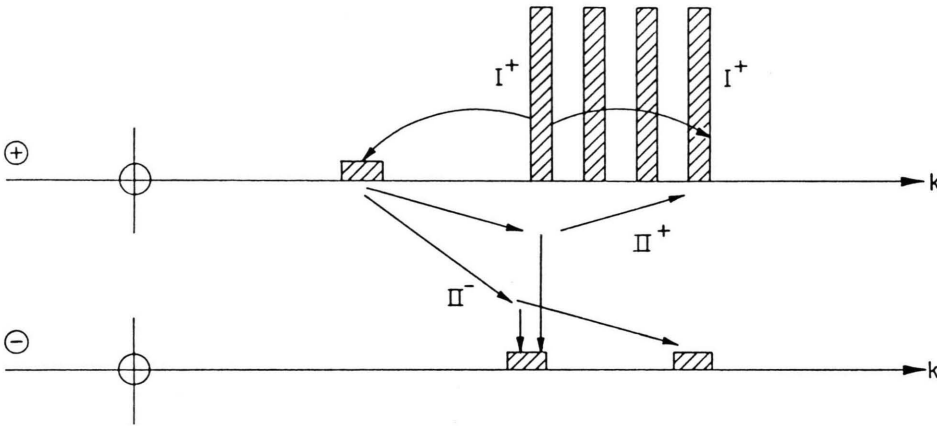
Fig. 9. A spatial distribution, consisting of only positive helicity, which may give rise to self-organization.

develop, the structure will become fuzzy and fade away. In the case described the helicity may be small.

In the second case we assume that the excited modes are all of one, say positive, helicity and that negative helicity modes are not excited. As in case one the small wave number region is void of energy. By means of the I^+ interaction, this region can be filled up, which corresponds to the evolution of structures in the turbulent medium. The energy transfer characteristic time is $\mathcal{O}(\varepsilon)$. This process is similar to the two-dimensional inverse cascade, however, in 3 D there are the two counteracting processes II^+ and II^- which tend to counteract the inverse cascade. The process II^- describes interaction of two negative and one positive helical modes which is of $\mathcal{O}(\varepsilon^2)$ because the negative helical modes are weakly excited. The interaction II^+ is also of order $\mathcal{O}(\varepsilon^2)$ as long as the amplitudes in the gap which is being filled up are still small. It increases however to $\mathcal{O}(\varepsilon)$ when the amplitudes in the gap have increased to $\mathcal{O}(1)$. We expect therefore an initial growth of spatially extended structures which after a while become fuzzy and decay. In addition, these transitions might be blocked by an appropriate structure of the spectral distribution. To elaborate this statement, consider interaction (4.12) and assume that $|a_k^+|$ and $|a_p^+|$ are $\mathcal{O}(1)$ and $|a_q^-| \sim 0$ and $k > p$. Differentiating the last equation and substituting the first two equations for $\dot{a}_{-k}^+$ and $\dot{a}_p^+$ gives

$$\ddot{a}_{-q}^- + \omega^2(t)\, a_{-q}^- = 0$$

with

$$\omega^2(t) = \tfrac{1}{4} |M|^2 (k-p)[(k+q)\,|a_k^+|^2 - (p+q)\,|a_p^+|^2].$$

The amplitude a_{-q}^- will grow exponentially only if $\omega^2(t) < 0$. The process will be stable if

$$|a_k^+|^2 \geqq \frac{p+q}{k+q}\, |a_p^+|^2, \quad \text{where} \quad \frac{p+q}{k+q} < 1, \quad p < k.$$

It thus appears that a spectrum of only positive and slightly decreasing helicity may be a relatively stable configuration. Recently Polifke and Shtilman [39] have performed a numerical integration of the Navier-Stokes equations on a $(64)^3$ grid. They compared strongly helical with nonhelical flow and found that large initial helicity impedes the transfer of energy towards smaller scales. However, for later times the flow does not remain maximally helical so that then the decay rate in the two cases is approximately equal. This behavior fits very well into our qualitative picture.

VI. Summary and Conclusions

We represented the hydrodynamic incompressible Euler equations by a set of base flows which are complete. Any possible incompressible flow can be expressed in the Hilbert space spanned by the base flows. We use eigenflows $\boldsymbol{v}_\lambda(\boldsymbol{r})$ of the curl operator in three dimensions such that $\boldsymbol{V} \times \boldsymbol{v}_\lambda(\boldsymbol{r}) = \lambda \boldsymbol{v}_\lambda(\boldsymbol{r})$. Each $\boldsymbol{v}_\lambda(\boldsymbol{r})$ is a stationary solution of the Euler equations. To each $\boldsymbol{v}_\lambda(\boldsymbol{r})$ corresponds another flow, $\boldsymbol{v}_{-\lambda}(-\boldsymbol{r})$, which is mirrored at the origin and which has a helicity of opposite sign. Thus the representation naturally splits into flows with positive and negative helicity.

From the Euler equations follow the equations of motion in 3 D for the coefficients of the eigenflows. They are nonlinear and can be divided into triplet interactions. We find interactions of modes of like helicity and of different helicity, which have the following properties:

Interactions of like helicity have an inverse cascade, i.e., the simultaneous conservation of energy and helicity enforces a flow of energy to smaller wave numbers if energy is to flow to larger numbers. The mode with the intermediate wave number is either the sole donor or acceptor of energy.

Interactions of mixed helicity need not display an inverse cascade. They can transport all energy in a transition to larger wave numbers, however, a partial migration to smaller wave numbers is also possible. One of the two modes of like helicity in a triplet is always the donor of energy. These relations are graphically represented in Fig. 7 and are the main result of this paper.

Our representation of the nonlinear interaction term $v \cdot \nabla v$ of the Navier-Stokes equation appears to be applicable to a variety of problems. For example, it is desirable to have a set of equations much simpler than the Navier-Stokes equations which describe the cascading of energy for turbulence in a driven system with dissipation. It is possible to derive such an equation from our representation in a fairly straightforward way (compare, e.g., Lee [40]).

Methods similar to the ones discussed here can be applied to the equations of magnetohydrodynamics. Frisch et al. [41] have shown that under certain conditions energy tends to accumulate in large structures of the magnetic field, using a method from equilibrium statistical mechanics for dissipationless systems. It would be desirable to see that this inverse cascade of the magnetic field can be deduced from the equations of motion directly, if they are cast into appropriate form. The inverse cascade of the magnetic field could then also be studied for dissipative systems.

Another problem is the study of hydrodynamic flow which is determined by boundary conditions, e.g., $v = 0$ on the boundaries of a region. In this case each of the base flows has to satisfy the boundary conditions. More and more base functions of the system can be added according to the level of excitation of the system. We will report about these topics in another place.

Appendix

The expression $(\hat{\boldsymbol{b}}_k + i\hat{\boldsymbol{k}} \times \hat{\boldsymbol{b}}_k)$ can be rotated around the $\boldsymbol{k}$-vector by an arbitrary phase factor without changing its magnitude. We use this property to calculate

$$M_{pq-k}^{\mathscr{E}_p \mathscr{E}_q \mathscr{E}_{-k}} = \boldsymbol{v}_p^{\mathscr{E}_p} \cdot (\boldsymbol{v}_q^{\mathscr{E}_q} \times \boldsymbol{v}_{-k}^{\mathscr{E}_{-k}})$$

by turning $\hat{\boldsymbol{b}}_p$, $\hat{\boldsymbol{b}}_q$, and $\hat{\boldsymbol{b}}_k$ into the plane defined by $\boldsymbol{p}$ and $\boldsymbol{q}$ (and $\boldsymbol{k}$). From Fig. 6 we obtain the components of the various vectors:

$$\hat{\boldsymbol{k}} = (1, 0, 0), \quad \hat{\boldsymbol{p}} = (\cos \alpha_q, \sin \alpha_q, 0),$$

$$\hat{\boldsymbol{q}} = (\cos \alpha_p, -\sin \alpha_p, 0),$$

$$\hat{\boldsymbol{b}}_k = (0, 1, 0), \quad \hat{\boldsymbol{b}}_p = (\sin \alpha_q, -\cos \alpha_q, 0),$$

$$\hat{\boldsymbol{b}}_q = (\sin \alpha_p, \cos \alpha_p, 0).$$

The triple product is now obtained by calculating $[(\boldsymbol{v}_p \times \boldsymbol{v}_q) \cdot \boldsymbol{v}_{-k}]$, where

$$\boldsymbol{v}_p = (\hat{\boldsymbol{b}}_p + i\hat{\boldsymbol{p}} \times \hat{\boldsymbol{b}}_p), \quad \boldsymbol{v}_q = (\hat{\boldsymbol{b}}_q + i\hat{\boldsymbol{q}} \times \hat{\boldsymbol{b}}_q),$$

$$\boldsymbol{v}_{-k} = (\hat{\boldsymbol{b}}_k - i\hat{\boldsymbol{k}} \times \hat{\boldsymbol{b}}_k).$$

The result is (4.6).

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